Spacetime line element

$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$

trajectories arbitrarily parameterized.

$x^\mu = (x^0, x^1, x^2, x^3) \Rightarrow x^\mu(\lambda) = (x^0(\lambda), \dots)$

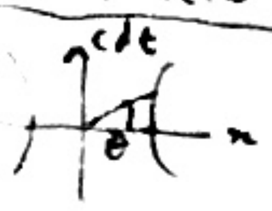
Spacetime interval

$s = \int_A^B \sqrt{|ds^2|} = \int_A^B d\lambda \sqrt{\pm\left(-\left(\frac{dx^0}{d\lambda}\right)^2 + \left(\frac{dx^1}{d\lambda}\right)^2 + \dots\right)}$

defining proper time $d\tau^2 = -\frac{ds^2}{c^2}$,

$\tau_{AB} = \int_A^B \sqrt{dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)} = \int_A^B \sqrt{1 - \frac{V(t')^2}{c^2}}\, dt'$

Angles/circles in spacetime



$ct = R\sinh\theta$

$x = R\cosh\theta$

vector and coordinate laws

If general coords x^μ then

$e_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}$, $g_{\mu\nu} = e_\mu \cdot e_\nu$

coord basis x^μ, $x^{\mu'}$

$e_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} e_\mu$ so $V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$

metric $g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$

Rapidities

define $v = \tanh\phi$.

$\gamma = \cosh\phi$, $\gamma v = \sinh\phi$

$= \sqrt{\frac{1}{1-v^2}}$ $= \sqrt{\frac{v}{1-v^2}}$

Rapidities add linearly:

$\phi_{12} = \phi_1 + \phi_2$

More Vector Laws

$F^{ab} = F_{cd}\,\eta^{ac}\eta^{bd} = \eta^{ac} F_{cd}\, \eta^{db}$

Kronecker Delta, Levi Civita

$\delta^\mu_\nu A^\nu = A^\mu$

$\delta^\mu_\sigma \delta^\sigma_\nu = \delta^\mu_\nu$

$\eta^{\mu\lambda}\eta_{\lambda\nu}\delta^\mu_\nu$

$\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\mu\rho\sigma}$

$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$

$(A \times B)_i = \epsilon_{ijk} A_j B_k$

$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$= \eta^{ab}\partial_a\partial_b$

Lorentz Transformations

$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow$ red. to Galilean $\frac{v}{c} \ll 1$.

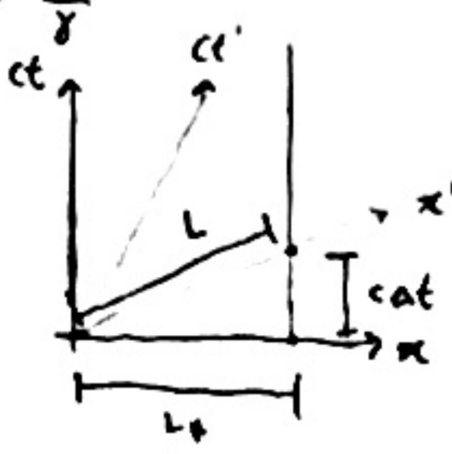
Time dilation

Particle moves at v rel to frame S. If τ is proper time, $t = \gamma\tau$ in frame S.

Length Contraction

object l_0 in rest frame. In frame moving at v,

$l = \frac{l_0}{\gamma}$



$L = L_x/\gamma$

Separation spacetime

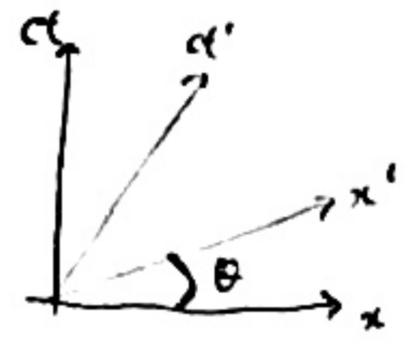
$\Delta s^2 > 0$ spacelike
$\Delta s^2 < 0$ timelike
$\Delta s^2 = 0$ null/lightlike

Lorentz Boost

$t' = \gamma\left(t - \frac{v}{c^2}x\right)$

$x' = \gamma(x - vt)$

$y' = y$ $z' = z$ for travelling frame of v.



frame S' moves at $v = c\tanh\theta$

Lorentz Transformation

$\Lambda^a_{\ b'} = \frac{\partial x^a}{\partial x^{b'}}$, and

$(\Lambda^{-1})^{b'}_{\ a} = \frac{\partial x^{b'}}{\partial x^a}$

for
$x^{0'} = \gamma(x^0 + vx^1)$
$x^{1'} = \gamma(x^1 + vx^0)$
$x^{2'} = x^2$
$x^{3'} = x^3$

$\Rightarrow \Lambda^a_{\ b'} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

clever vector tricks

decompose p^μ into parallel/orthog to u

$p^\mu = -(p\cdot u)u^\mu + p^\mu_\perp$

Four-vectors

$a = (a^0, \vec{a}) = a^\alpha e_\alpha$

$a \cdot b = \eta_{\alpha\beta} a^\alpha b^\beta$

$\eta_{\alpha\beta} = e_\alpha \cdot e_\beta$ so

$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$

Four-velocity

$u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{1}{\gamma}, \frac{\vec{v}}{\gamma}\right)$

$u \cdot u = u^\mu u_\mu = -1$

unit timelike, parallel to world line.

Velocity Addition

frame S particle vel $\vec{v}$, S' moving at v along x,

$v^{x'} = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{v^x - v}{1 - v\frac{v^x}{c^2}}$

$v^{y'} = \frac{v^y}{1 - v\frac{v^x}{c^2}}\frac{1}{\gamma}$, sim for $v^{z'}$

Kinematic Laws

$p^\mu = mu^\mu = (\gamma m, \gamma m v_x, \dots)$

$= (E, \vec{p})$ for real forces.

$a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$, $f^\mu = ma^\mu$

$\vec{a} = \frac{d\vec{p}}{dt}\frac{1}{m}$, $\vec{F} = \frac{d\vec{p}}{dt}$, $a \cdot u = 0$, $f \cdot u = 0$

$f^\mu = (\gamma\vec{F}\cdot\vec{v}, \gamma\vec{F})$

$f^\mu f_\mu = m^2a^2$ frame-invariant, sim. for a

$P^\mu_{tot} = \text{constant}$ conservation of four-momentum

$p^\mu p_\mu = -m^2$ Always

For 2 Lorentz Transformations

If frame B moves at $v_{B/A}$ rel to A and frame C moves at $v_{C/B}$ rel to B

$\gamma_{C/A} = \gamma_{C/B}\gamma_{B/A}(1 + v_{B/A}v_{C/B})$

Taylor Expansions

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$(1+\beta x)^\alpha \approx 1 + \alpha\beta x + O(\beta x)^2$

Variational Principle (Euler-Lagrange)

$S[x(t)] = \int_{t_A}^{t_B} L(\dot{x}(t), x(t))\, dt$

extremum of $f(x) \Rightarrow \delta f = \sum_{a=1}^{n} \frac{\partial f}{\partial x^a}\delta x^a = 0$

$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0$

example $L = \frac{1}{2}m\dot{x}^2 - V(x)$

Electrodynamics

We combine $\vec{B} = \nabla\times\vec{A}$ and $\vec{E} = -\nabla\phi$ to four-potential,

$A^\mu = (\phi, \vec{A})$

Faraday Tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$

under transformation:

$F^{\mu'\nu'} = \Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\mu\nu}$

or

$A^{\mu'\nu'} = \Lambda^\mu_{\ \nu}A^{\mu\nu}$ then

$F^{\mu'\nu'} = \partial^{\mu'}A^{\nu'} - \partial^{\nu'}A^{\mu'}$

Defining $j^\mu = (\rho, \vec{j})$ charge density four-vector

$\nabla \cdot B = 0$, $\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \Rightarrow \epsilon_{\alpha\beta\gamma\delta}\partial^\alpha F^{\beta\gamma} = 0$

$\nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$, $\nabla\cdot\vec{E} = \rho \Rightarrow \partial_\alpha F^{\alpha\beta} = -j^\beta$

All tensors w/fully contracted indices frame-inv

$\frac{1}{2}F^{ab}F_{ab} = \vec{E}^2 - \vec{B}^2$, $\frac{1}{8}\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} = \vec{E}\cdot\vec{B}$

Hyperbolic Trig properties

$\cosh^2 x - \sinh^2 x = 1$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$, $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$

$\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$, $\int \tanh x = \ln(\cosh x)$

for any transform:

$dx = \frac{dx}{dt}dt + \frac{dx}{dy}dy + \dots$

$dx^a = \frac{dx^a}{dx^n}dx^n$

To prove vector invariance

$F^{\mu\nu} \mapsto (\Lambda^{-1})^\rho_{\ \mu}(\Lambda^{-1})^\sigma_{\ \nu}F_{\rho\sigma}$ Apply arbitrary transforms

$F^{\mu\nu} \mapsto \Lambda^\mu_{\ \rho}\Lambda^\nu_{\ \sigma}F^{\rho\sigma}$

$\epsilon_{\mu\nu\rho\sigma} \mapsto (\Lambda^{-1})^\alpha_{\ \mu}(\Lambda^{-1})^\beta_{\ \nu}(\Lambda^{-1})^\gamma_{\ \rho}(\Lambda^{-1})^\delta_{\ \sigma}\epsilon_{\alpha\beta\gamma\delta}$