Dynamics - Newtonian Mechanics

$$m\ddot{x} = F(t, x, v) = \frac{d\vec{p}}{dt}$$

for $F(t)$: $m\frac{dv}{dt} = F(t)$

$$m\int_{v_0}^{v(t)} dv' = \int_{0}^{t} F(t')\,dt'$$

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\,dt'$$

$$v(t) = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t)\,dt'$$

$$x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'$$

for $F(x)$: $mv\frac{dv}{dx} = F(x)$

$$\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

$m\int v\,dv = \int F(x)\,dx$ so

$\frac{1}{2}mv^2 = \int F(x)\,dx$

Kinetic energy = Work Done

$F = -\frac{dV}{dx} \Rightarrow \frac{1}{2}mv^2 + V(x) = c$

$$m\int_{v_0}^{v(x)} v'\,dv' = \int_{x_0}^{x} F(x')\,dx' \Rightarrow \int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt'$$

typically LHS integral not doable.

for $F(v)$: $m\frac{dv}{dt} = F(v)$

$$m\int_{t_0}^{t} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt' \quad \text{so}$$

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\,dt'$$

$\vec{p} = m\vec{v}$, $\frac{d}{dt}\sum_i p_i = 0$ Conservation of Momentum.

rotations: $\vec{L} = \vec{r}\times\vec{p}$ (Ang. Moment.), $\vec{N} = \vec{r}\times\vec{F} = \dot{\vec{L}}$, $\vec{v} = \vec{\omega}\times\vec{r}$

$\frac{d\vec{L}}{dt} = \vec{N}_{ext}$, if $\vec{N}_{ext} = 0$, $\vec{L}$ is const (across Galilean trans.)

Work-energy: $E = T + U$

$W = \int \vec{F}\cdot d\vec{r}$, $T = \frac{1}{2}mv^2$, $W_{net} = \Delta T$

if F conservative, $\vec{F} = -\nabla U$, $W_{12} = -(U_2 - U_1)$

if only conservative forces act: E = const.

Diff Eq Linearization

recall Taylor Series Definition:

$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}$ similarly,

$\vec{f}(\vec{x}) = H(x_0)\vec{x} + O(\vec{x}^2)$ Hessian.

$$H_{ij} = \frac{\partial f_i}{\partial x_j} \Rightarrow \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \cdots \end{pmatrix}$$

so linearized EoM is

$m\ddot{x}_i = H_{ij}x_j$

David Kim

Oscillation

Simple Harmonic oscillator: $m\ddot{x} + kx = 0$

often rewritten $\ddot{x} + \omega_0^2 x = 0$, $\omega_0 = \sqrt{\frac{k}{m}}$

$x(t) = A\cos(\omega_0(t - t_0) + \phi)$ general soln

$= A\cos(\omega_0(t-t_0)) + B\sin(\omega_0(t-t_0))$

$v(t) = -A\omega_0\sin(\omega_0 t + \phi)$, $a(t) = -\omega_0^2 x(t)$

Period: $T = \frac{2\pi}{\omega_0}$, $f = \frac{1}{T}$, $\omega_0 = 2\pi f$

Damped oscillator: $m\ddot{x} + b\dot{x} + kx = 0$ rewritten to std form

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, $\beta = \frac{b}{2m}$

underdamped: $\beta < \omega_0$

$x(t) = Ae^{-\beta t}\cos(\omega_d t + \phi)$, $\omega_d = \sqrt{\omega_0^2 - \beta^2}$

critically damped: $\beta = \omega_0$

$x(t) = (C_1 + C_2 t)e^{-\beta t}$ no real oscillation, returns to equilibrium ASAP w/o overshoot.

overdamped: $\beta > \omega_0$

$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ negative real roots.

$r_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

for damping $F_d = -b\dot{x}$, $P = -b\dot{x}^2$ so $\frac{dE}{dt} < 0$.

quality factor $Q = \frac{\omega_0}{2\beta}$ for weak damping.

Driven oscillator: $m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega t)$

Steady-state soln: $x(t) = A(\omega)\cos(\omega t - \delta)$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}, \quad \text{phase lag } \tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

$\omega \ll \omega_0$: $\delta \approx 0$ (low-freq), $\omega \approx \omega_0$: $\delta \approx \frac{\pi}{2}$ (resonance), $\omega \gg \omega_0$: $\delta \approx \pi$ (high-freq)

Resonance - drive freq very near natural frequency.

for weak damping: $\omega_r = \sqrt{\omega_0^2 - 2\beta^2}$

Steady-state: After long time, transient (init conds) dies out long term.

Driven not damped: $\ddot{x} + \omega_0^2 x = \frac{f(t)}{m}$

IN GENERAL: $x(t) = x_{hom}(t) + x_{part}(t)$

for forced (undamped): $f(t) = F_0\cos(\omega t)$

solve homogenous $\ddot{x} + \omega_0^2 x = 0$, $x_h = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

then try ansatz $x_p = A\cos(\omega t)$:

$A(\omega_0^2 - \omega^2)\cos(\omega t) = \frac{F_0}{m}\cos(\omega t) \Rightarrow A = \frac{F_0/m}{\omega_0^2 - \omega^2}$

if $\omega \to \omega_0$, $A \to \infty$, energy builds up, large oscill.

$\omega \ll \omega_0$, system follows force slowly

$\omega \gg \omega_0$, system can't keep up,

for driven + damped:

solve hom. $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, $x_h = e^{-\beta t}(A\cos\omega_d t + B\sin\omega_d t)$

ansatz $x_p = A(\omega)\cos(\omega t - \delta)$

FOR ARBITRARY $f(t)$:

Fourier: $f(t) = \sum f_n\cos(n\omega t)$ solve for all parts of the driving force then add.

Green's: decompose $f(t) = \int f(t')\delta(t - t')\,dt'$

defining $y(t) = x(t) - x_0 \Rightarrow \ddot{y} + \omega_0^2 = \frac{f(t)}{m}$ just dsp. from equilibrium

define $G(t, t')$ response at t to sharp unit impulse at prev. time t' $\Rightarrow \ddot{G}(t,t') + \omega_0^2 G(t,t') = \delta(t - t')$

impose boundary conditions: $G(t, t') = 0$ for $t < t'$

$$\int_{t'-\epsilon}^{t'+\epsilon} (\ddot{G} + \omega_0^2 G)\,dt = \int_{t'-\epsilon}^{t'+\epsilon} \delta(t - t')\,dt \rightarrow \begin{aligned} &G(t'^+, t') - G(t'^-, t') = 0 \\ &\dot{G}(t'^+, t') - \dot{G}(t'^-, t') = 1 \end{aligned}$$

$\Downarrow$ $= 1$

$\Rightarrow G(t', t') = 0$, $\dot{G}(t', t') = 1$

Gravitation

$F(r) = -\frac{GMm}{r^2}$, $\vec{F}(\vec{r}) = -G\frac{M_1 M_2}{r^2}\hat{r}$, $U(r) = -G\frac{Mm}{r}$

$\vec{g}(\vec{r}) = \frac{\vec{F}}{m}$, $\vec{g} = -G\frac{M}{r^2}\hat{r}$ field for a point mass, $\Phi(r) = \frac{U}{m} = -G\frac{M}{r}$

$F = -\nabla U$, $\vec{g} = -\nabla\Phi$, $W_{12} = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \int_{r_1}^{r_2} -\frac{GMm}{r^2}\,dr$

escape velocity: $\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0 \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$

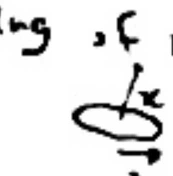
circular orbit: $\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow v = \sqrt{\frac{GM}{r}}$, $\omega = \frac{v}{r} = \sqrt{\frac{GM}{r^3}}$, $T = 2\pi\sqrt{\frac{r^3}{GM}}$

$T = \frac{1}{2}mv^2 = \frac{GMm}{2r}$, $U = -\frac{GMm}{r} \Rightarrow E_{tot} = -\frac{GMm}{2r}$

$L = \sqrt{R^2 + r^2 - 2rR\cos\theta}$, area of ring $d\Omega = 2\pi R^2\sin\theta\,d\theta$

mass density $\sigma = \frac{M}{4\pi R^2}$, $d\Phi = -\frac{Gm\sigma\,2\pi R^2\sin\theta\,d\theta}{L}$ at point o

Continuous Mass dists: $d\vec{g} = -G\frac{dm}{r^2}\hat{r}$, $d\Phi = -G\frac{dm}{r}$

ring of mass:  $r = \sqrt{x^2 + a^2}$, $\Phi(x) = -G\frac{M}{\sqrt{x^2 + a^2}}$, $g_x = -\frac{d\Phi_x}{dx} = -G\frac{Mx}{(x^2 + a^2)^{3/2}}$

disk: $\sigma = \frac{M}{\pi R^2}$ so $dm = \sigma\,dA = \sigma(2\pi s\,ds)$.

$d\Phi = -\frac{G(2\pi s\,ds)}{\sqrt{x^2 + s^2}}$, $\Phi(x) = -2\pi G\sigma\int_0^R \frac{s\,ds}{\sqrt{x^2 + s^2}} = -2\pi G\sigma\left(\sqrt{x^2 + R^2} - x\right)$

$g_x = -2\pi G\sigma\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$

rod: $dm = \lambda\,dx$, $\lambda = \frac{M}{L}$

on rod axis: $r = a - x$, $d\Phi = -\frac{G\lambda\,dx}{a - x}$, $\Phi(a) = -G\lambda\ln\left(\frac{a}{a - L}\right)$

Perp from center: $r = \sqrt{x^2 + y^2}$, $d\Phi = -\frac{G\lambda\,dx}{\sqrt{x^2 + y^2}}$, $\Phi = -G\lambda\ln\left(\frac{L/2 + \sqrt{(L/2)^2 + y^2}}{-L/2 + \sqrt{(L/2)^2 + y^2}}\right)$

infinite. $g_y = -\frac{G\lambda L}{y\sqrt{(L/2)^2 + y^2}} \Rightarrow g_y = \frac{2G\lambda}{y}$

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if we assume $f(t)$ even, $f(t) = \sum_{n=0}^{\infty} f_n\cos(n\omega t)$ and so

$f_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t) \Rightarrow x_n(t) = A_n\cos(n\omega t - \delta_n)$

$$A_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2}}, \quad \delta_n = \arctan\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$

$$\langle x^2\rangle = \frac{1}{T}\int_{-T/2}^{T/2} x^2\,dt = \frac{1}{T}\int_{-T/2}^{T/2} A_n A_m\cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)\,dt$$

$= \begin{cases} \frac{T}{2} & m = n \neq 0 \\ T & m = n = 0 \\ 0 & m \neq n \end{cases}$

$\Rightarrow$ for this case, $\ddot{G} + \omega_0^2 G = 0$ so

$G(t, t') = A\cos(\omega_0(t - t')) + B\sin(\omega_0(t - t'))$ with B.C.s $A = 0$, $B = \frac{1}{\omega_0}$

$= \frac{1}{\omega_0}\sin(\omega_0(t - t'))\Theta(t - t')$ and

$$y(t) = y_h(t) + \frac{1}{m\omega_0}\int \sin(\omega_0(t - t'))f(t')\,dt'$$

More Oscillation

equilibrium: $F(x_0) = 0$

let $x = x_0 + \delta \Rightarrow F(x) = F(x_0) + (x - x_0)F'(x_0) + \dots$
$= F'(x_0)(x - x_0)$

$k \doteq -F'(x_0)$, harmonic oscillator!

$F(x) = -k(x - x_0)$, $m\ddot{x} = k(x - x_0)$

$\ddot{x} + \omega^2(x - x_0) = 0$, $\omega^2 = \frac{k}{m}$

stable if $F'(x_0) < 0 \to F = -k(x - x_0)$
unstable if $F'(x_0) > 0 \to F = +k(x - x_0)$

$F = \frac{dU}{dx} \Rightarrow$ equil @ $\frac{dU}{dx} = 0$

$U(x) \approx U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2$ so

$F = -U''(x_0)(x - x_0)$, $m\ddot{x} + U''(x_0)(x - x_0) = 0$

$\omega^2 = \frac{U''(x_0)}{m}$, $k = U''(x_0)$

$U'' > 0$: min, stable. $U'' < 0$: max, unstable.

normal modes (many oscillators):

$m_i\ddot{x}_i = F_i(x_1, \dots, x_N)$, $F_i(\vec{x}) = \sum_j \frac{\partial F_i}{\partial x_j}\Big|_0 x_j$

define stiffness $K_{ij} = -\frac{\partial F_i}{\partial x_j}\Big|_0$, $F_i = \sum_j k_{ij} x_j$

then $M\ddot{\vec{x}} + K\vec{x} = 0$, $M = \mathrm{diag}(m_1, \dots, m_N)$

Ansatz: $\vec{x} = \vec{v}e^{i\omega t}$, $\ddot{\vec{x}} = -\omega^2\vec{v}e^{i\omega t}$ plugging in,

$(-\omega^2 M + K)\vec{v} = 0 \Rightarrow K\vec{v} = \omega^2 M\vec{v}$

eigenvalue problem! ω^2 eigenvals, $\vec{v}$ eigenvect

for nontrivial: $\det(K - \omega^2 M) = 0$ determines allowed ω^2

now we can write $\vec{x} = \sum_n q_n(t)\vec{v}_n$ where

$\ddot{q}_n + \omega_n^2 q_n = 0$ superposition of many modes (harmonic oscill.)

$\vec{x}(t) = \sum_n [A_n\cos(\omega_n t) + B_n\sin(\omega_n t)]\vec{v}_n$

SHO

given initial position, velocity (x_0, v_0),

$x(t) = x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$, $A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$ amplitude max disp.

Moving Support

one end of spring/pendulum moves as $y(t)$.

$$m\ddot{x} + m\beta\dot{x} + m\omega_0^2 x = \underbrace{F(t)}_{\text{ext. drive}} + \underbrace{m\omega_0^2 y(t)}_{\text{support moving.}}$$

similar to external drive.

Pendulum

mass m, length l, angle θ.

$\tau = -mgl\sin\theta$, $I\ddot{\theta} = \tau$ for point mass $I = ml^2$

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$ small angle approx.

$\omega_0 = \sqrt{\frac{g}{l}}$, $T = 2\pi\sqrt{\frac{l}{g}}$, $\theta(t) = A\cos(\omega_0 t + \phi)$

$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta)$

Math, vectors, misc. properties

$\hat{A} = \frac{\vec{A}}{|A|}$, $\vec{A}\cdot\vec{B} = AB\cos\theta$, $|\vec{A}\times\vec{B}| = AB\sin\theta$

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$, $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D})$

coord transform $x_i' = \sum_j \Lambda_{ij}x_j$

rotation matrices: $\Lambda^T\Lambda = I$, $\Lambda^T = \Lambda^{-1}$

proper rot $\det(\Lambda) = +1$, improper rot $\det(\Lambda) = -1$

rotation about z: $\Lambda = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}}$, $\frac{d}{dt}(\vec{A}\times\vec{B}) = \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}$

Cylindrical coord transform:
$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$, $\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$

Spherical coord transform:
$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$

divergence thm: $\int_S \vec{A}\cdot d\vec{a} = \int_V (\nabla\cdot\vec{A})\,d\tau$

Stokes thm: $\oint_C \vec{A}\,d\vec{s} = \int_S (\nabla\times\vec{A})\cdot d\vec{a}$

Gravitation

$\Phi(\vec{r}) = -G\int d^3r'\rho(r')\frac{1}{|\vec{r} - \vec{r}'|}$.

$g(\vec{r}) = -G\int d^3r'\rho(r')\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = -\nabla\Phi(\vec{r})$

$\int_S \vec{g}\cdot d\vec{a} = -4\pi G M_{enc}$, $M_{enc} = \int_{V, \partial V = S} \rho(r')\,d^3r'$

Green's funct.

rewrite system as $Lx(t) = f(t)$

then green's funct $LG(t, t') = \delta(t - t')$

$x(t) = \int G(t, t')f(t)\,dt$

for SHO: $L = \frac{d^2}{dt^2} + \omega^2$

damped: $L = \frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega^2$

$\left(\frac{d^2}{dt^2} + \omega_0^2\right)G(t, t') = \delta(t - t')$

then $G(t, t') = A\sin\dots$

Taylor exp.

$\sin x = \sum_{n=0} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

$\cos x = \sum_{n=0} \frac{(-1)^n}{(2n)!}x^{2n}$

$e^x = \sum_{n=0} \frac{x^n}{n!}$

$(1 + \beta x)^\alpha \approx 1 + \alpha\beta x + O(\beta x)^2$

Known Integrals

$\int \frac{1}{\sqrt{1 - x^2}}dx = \sin^{-1}x$

$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

$\int \ln x\,dx = x\ln x - x$

$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$

$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x\cos x)$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right)$

$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right)$

subs: $a^2 - x^2 \to x = a\sin\theta$
$a^2 + x^2 \to x = a\tan\theta$
$x^2 - a^2 \to x = a\sec\theta$

Trig Properties

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$

$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$

$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a\tan b}$

$\sin 2x = 2\sin x\cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin a\sin b = \frac{1}{2}[\cos(a - b) - \cos(a + b)]$

$\cos a\cos b = \frac{1}{2}[\cos(a - b) + \cos(a + b)]$

$\sin a\cos b = \frac{1}{2}[\sin(a + b) + \sin(a - b)]$

$A\sin(\omega t) + B\sin(\omega t) = C\cos(\omega t - \phi)$
$\hookrightarrow C = \sqrt{A^2 + B^2}$, $\tan\phi = \frac{B}{A}$

Fourier Series

periodic $f(x)$ w/ period 2π:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = \int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$

$\int_{-\pi}^{\pi}\sin(nx)\cos(mx) = 0$. for general $[-L, L]$, replace $x \to \frac{n\pi x}{L}$

Square wave: $f(x) = \sum_{n\text{ odd}} \frac{1}{n}\sin(nx)$

Sawtooth: $f(x) = \sum \frac{(-1)^{n+1}}{n}\sin(nx)$